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Exact solutions of hydrodynamics and their applications in heavy-ion physics

PhD Thesis

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1 Introduction

According to our present understanding, the Universe has formed in a hot and dense state (after an initial exponential expansion, which is called *inflational period*), and then expanded and cooled down. After a series of phase transitions, matter has taken the shape we can observe today. Nowadays it is possible to experimentally investigate one of these phase transitions, the deconfinement transition of Quantum Chromodynamics (QCD). Heavy-ion experiments serve this purpose. Accelerators (such as SPS, RHIC, LHC) collide „heavy ions" (ie. heavy atoms with all electrons stripped away) to create a hot and dense matter that quickly expands and cools, and as it reaches the detectors situated around the interaction point, it is again just a set of familiar particles that are well known from elementary particle reactions. However, the multiplicity and momentum distributions, and various correlations of these particles carry information about the matter created in the collision.

One of the focal questions related to heavy-ion physics was if a predicted state of matter called „quark gluon plasma" [d1], a state consisting of deconfined quarks and gluons really forms at high enough temperatures and pressures. This question got answered only recently as it turned out that at sufficiently high collision energy, a medium with quark and gluon degrees of freedom forms, however, instead of behaving as a gas of freely roaming quarks and gluons, as expected, it rathert resembles a strongly coupled liquid [d2]. This statement is based on the fact that although many types of models attempted to describe the collective properties of this matter, for among them, models based on hydrodynamics were the most successful. The appearance of this „perfect fluid of quarks" has been, for the first time, demonstrated unambiguously at the RHIC accelerator (BNL, USA); at higher collision energies, results from the LHC accelerator show a qualitatively similar picture of the matter created in heavy-ion collisions [d3].

Hydrodynamical models treat the created matter as an expanding fireball, and model the particle production based on locally thermalized distributions. Hydrodynamics can be successfully applied to a wide range of phenomena: such as the expansion of the whole Universe, the everyday aspects of fluid motion as well as the description of heavy-ion reactions; the ultimate reason behind this is that the basic hydrodynamical equations do not contain a physical scale, they only express local conservation of charge, energy and momentum, and local thermal equilibrium. Although it averages over some interesting microscopic degrees of freedom, hydrodynamics gives new insights, too: knowing the al state and the equations of state, we can connect initial conditions to the final state and most importantly to experimental observables. Moreover, collective properties such as speed of sound, equation of state, viscosity yield information on the microscopic nature of the matter, so they are interesting on their own right.

On the other hand, the hydrodynamical equations are complicated nonlinear partial differential equations, their solutions are, most frequently, obtained only numerically. Nevertheless, theoretically it is very interesting and challenging to find their exact solutions: in case of nonlinear equations such solutions always give insight in the structure and nature of the equations. It must also be noted that such solutions and models based on them can directly be applied to the description of experimental data in many cases.

There was a general expectation in the phenomenology of heavy ion collisions that models that are based on binary collisions of elementary particles might yield a good description of the observables. However, the simple scaling properties of the data led to the unexpected result that simple analytic solutions of hydrodynamics which describe exploding fireballs grasp the essence of the dynamics of heavy-ion collisions[1] [d4]. If such solutions contain free parameters with physical meaning, then the determination of them from the data leads to an understanding of the space-time evolution of the reaction.

2 Aims and methods of this work

The mentioned facts motivated me to begin work on the search for exact solutions of hydrodynamics. These may give the relation between the initial and final state in a simple analytic form, in contrary to the (in other sense very useful) numerical models. In this research I followed the following goals (formulated together with my supervisor):

1. Search for simple solutions of relativistric hydrodynamics that are exact, analytic and realistic (ie. qualitatively similar to the space-time picture of high-energy heavy-ion reactions). There were many known expanding fireball-like nonrelativistic hydrodynamical solutions [d5,d6], and models based on nonrelativistic hydrodynamics were surprisingly useful in the description of the data. (An example is the interpretation of the mass scaling of the effective temperature of particle spectra [d7]). Nevertheless, for some observables relativistic kinematics has substantial role (such as the so-called rapidity distribution or the hydrodynamical description of the rapidity dependent azimuthal anisotropy [d8]). The description of rapidity distributions prompted the first most important exact relativistic hydrodynamical solutions [d9], and there was a class of self-similar solutions known [d10]. The natural question arose if one can find similar simple solutions that feature for example non-zero acceleration.

[1] This does not mean that the expansion of the fireball formed in heavy-ion collisions is mathematically equivalent to some exact analytic hydrodynamical solution, but it does mean that such solutions are very good approximations of the real dynamics, which is a result of essential importance.

2. The investigation of the relation between the collisionless kinetic equation and the hydrodynamical equations. One of the interesting known nonrelativistic hydrodynamical solution was discovered by the realization that the phase-space distribution of a special hydrodynamical parametrization has a property that if one lets it evolve without collisions, it generates a solution to the hydrodynamical equations too [d6]. This means that for special microscopic initial conditions the collisionless motion of the particles of the fluid the local thermal equilibrium is retained. The question was if there exist other such, perhaps not yet known solutions with this property, and that whether this idea can be generalized to relativistic case. It is also an interesting question that what is the effect of this „collisionlessness" on the observables.

3. Exact or semi-analytic calculation of observables. I have attempted to describe observables whose behaviour was not yet fully understood, in terms of new hydrodynamical solutions. The observable of choice was the rapidity distribution of the produced particles, which is connected to the energy density of the reaction.

In the search for new solutions, one can apply numerical methods and perturbative expansion around known simple solutions. However, the most efficient method was the direct investigation of the basic equations: many different transformations and tricks were needed. In comparison, the mentioned notion of „collisionlessness" is essentially very simple to investigate (although as far as I know, it was not done before for the relativistic case): one needed only some simple transformations of the coupled Boltzmann equation and the hydrodynamical equations. This method led to new, rotating exact solutions as well (detailed in the Theses below).

To compute the rapidity distribution of the produced particles, I used analytical integral formulas, then numerically evaluated them, and applied an approximation based on saddle-point integration as well: the error of this turned out not to be significant (from the comparison with the numerical result). Thus for fitting the rapidity distribution measured by the BRAHMS collaboration, I used the analytical approximation[2]. The results made it possible to improve the Bjorken estimate, a widely used method to infer the initial energy density, by a roughly 100% correction.

3 Theses — my contributions

1. As a generalization of the Hwa-Bjorken hydrodynamical solution, I have found the first class of exact solutions of relativistic hydrodynamics which fits to the phenomenology of high-

[2] For the comparison with experimental data, for fitting and plotting I used the Minuit library as well as the CERN ROOT package.

energy reactions, and where the thermodynamical quantities and the velocity have simple and explicit expressions, while the solution also feautures a non-vanishing acceleration. Such solutions were not known before. As solutions to a complicated nonlinear system of differential equations, these are interesting on their own, in mathematical physical sense. One of their possible applications is testing numerical hydrodynamical methods [a1,a2].

2. I have investigated the coupled relativistic kinetic theory and the hydrodynamical equations and have proven that there exist flow and temperature profiles that are solutions of relativistic hydrodynamics, while the thermal phase space distribution corresponding to them satisfies the collisionless Boltzmann equation. In these cases the local thermal distribution of the particles participating in the flow motion can be retained without any reference to microscopic collisions, even in the case of accelerating collective motion. I have found all possible relativistic hydrodynamical solutions with this property [a3].

3. As generalizations of the new collisionless solutions I have found the first solutions of relativistic hydrodynamics which are rotating, thus are beyond spherical symmetry. I have found an other class of realistic exact solutions which are able to describe systems with finite spatial size. Rotating flows are important because in non-central heavy-ion collisions, the created matter has non-vanishing angular momentum, and the solutions I have found may be able to describe the expansion of such fireballs, and to investigate the role of the impact parameter analytically [a3].

4. In $\sqrt{s_{NN}} = 200$ GeV center-of-mass energy gold-gold collisions at RHIC, I have modelled the longitudinal collective flow with one of our newly found parametric accelerating exact solutions, and I have calculated the rapidity and pseudorapidity distribution of charged hadrons produced in the reaction, both numerically and with an analytic approximation. Comparing this result to experimental measurements by the BRAHMS collaboration, I have concluded that this simple model can be fit to the measured data: taking acceleration into account makes it possible to describe the finite width of the distribution. From the fit I have determined the characteristic parameter of the longitudinal acceleration. As an application of this result, I substantially improved the estimation of the initial energy density in heavy-ion collisions. Taking the acceleration of the flow (ie. the work done during the acceleration) into account, with the new estimation I have obtained 10 ± 0.5 GeV/fm^3 for the maximal, $\sqrt{s_{NN}} = 200$ GeV energy Au+Au collision energy of the RHIC accelerator (compared to the Bjorken estimate of 5 GeV/fm^3). With a conjecture for general equations of state I have obtained a further 50% increase in the initial energy density compared to this advanced estimation [a1,a2].

4 Conclusion and outlook

The search for exact solutions of hydrodynamics is an exciting theoretical challenge; many different attempts were needed. In my thesis I summarized only those attempts that turned out to be successful. The classes of solutions described here are mature and a settled topic in many sense. The results of the thesis illuminate the statement that besides numerical solutions, exact solutions do have a role in collective models in heavy-ion physics, and they are worth studying. The newly found exact relativistic solutions can be also useful in other branches of physics (such as astrophysics and cosmology).

Knowing the initial energy density is of utmost importance in the theoretical interpretation of heavy-ion experimental results, that is, to determine the position of experimental data on the temperature-baryochemical potential diagram. The presented results on the energy density (published in 2008) are in accordance with PHENIX experimental results (published later in 2010) that determined the temperature of the quark-gluon-plasma formed in Au+Au collisions based on the spectrum of direct photons. My results show that the initial energy density expected for LHC, $\simeq$ 15 GeV/fm^3 is already reached at RHIC collision energies. We can conclude that at LHC the initial energy density will be even higher than expected. Concerning RHIC, we can reinforce the conclusion based on the Bjorken estimate: the energy density is far above the phase transition value of $\simeq$ 1 GeV/fm^3, based on lattice QCD calculations.

The generalizations of the new exact solutions (which might need some fundamentally new methods) cound be of great interest. For example, it would be very much interesting to find exact accelerating relativistic solutions with more general, ellipsoid-like symmetry as this fits to the geometry of a heavy-ion collision. Other, more general equations of state should be investigated too. The question if there exists viscous, non-ideal fluid generalizations of the presented solutions leads to a completely new research topic.

5 Publications

5.1 Papers which the theses are based on

a1. T. Csörgő, M. I. Nagy, M. Csanád: *A New family of simple solutions of perfect fluid hydrodynamics*, Phys. Lett. B **663**, 306 (2008), arXiv:nucl-th/0605070.

a2. M. I. Nagy, T. Csörgő, M. Csanád: *Detailed description of accelerating, simple solutions of relativistic perfect fluid hydrodynamics*, Phys. Rev. C **77**, 024908 (2008), arXiv:07093677 [nucl-th].

a3. M. I. Nagy: *New simple explicit solutions of perfect fluid hydrodynamics and phase-space evolution*, Phys. Rev. C **83**, 054901 (2011), arXiv:09094285 [nucl-th].

5.2 Other papers connected to the topic of the thesis

b1. M. I. Nagy:*QCD EoS, initial conditions and final state from relativistic hydrodynamics in heavy-ion collisions*, In: Proceedings of the 38th International Symposium on Multiparticle Dynamics (ISMD08), DESY-PROC-2009-01 (2009), arXiv:0902.0377 [hep-ph].

b2. M. Csanád, M. I. Nagy and T. Csörgő: *Similar final states from different initial states using new exact solutions of relativistic hydrodynamics*, Eur. Phys. J. ST **155**, 19 (2008), arXiv:0710.0327 [nucl-th].

b3. T. Csörgő, M. I. Nagy and M. Csanád: *New exact solutions of relativistic hydrodynamics*, J. Phys. G **35**, 104128 (2008), arXiv:0805.1562 [nucl-th].

b4. T. Csörgő, M. I. Nagy and M. Csanád: *Accelerating Solutions of Perfect Fluid Hydrodynamics for Initial Energy Density and Life-Time Measurements in Heavy Ion Collisions*, Braz. J. Phys. **37**, 723 (2007), arXiv:nucl-th/0702043.

5.3 Selected talks on the topic of this thesis

c1. *New exact solutions of perfect fluid hydrodynamics and an advanced estimate of the initial energy denstiy and the life-time of the reaction*, NCRH2007 workshop, Frankurt, Apr 16 2007.

c2. *QCD EoS, initial conditions and hydrodynamical final state in heavy-ion collisions*, ISMD 2008, Hamburg, Germany, Sep 16, 2008.

c3. *Hydrodynamical solutions: theory and applications in high-energy physics* (in Hungarian), Nuclear Physicists' meeting, Jávorkút, Sep 4, 2009.

c4. *Simple explicit solutions of perfect fluid hydrodynamics and phase-space evolution*, Zimányi 2009 Winter School on Heavy Ion Physics, Budapest, Dec 3, 2009.

c5. *New exact solutions of perfect fluid hydrodynamics: theory and applications in high-energy experiments*, Gribov-80 Memorial Workshop, Trieste, Italy, May 27, 2010.

c6. *Recent exact solutions of perfect fluid hydrodynamics*, Zimányi 2011 Winter School on Heavy Ion Physics, Budapest, Nov 30, 2011.

c7. *Application of exact hydrodynamical solutions in heavy ion phenomenology* (in Hungarian), ELTE Particle Physics Seminar, Mar 28, 2012.

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B. B. Back *et al.* [PHOBOS Collaboration], Nucl. Phys. A **757**, 28 (2005).

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